

Another introduction to the geometry of metric spaces

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Abstract

Here Lipschitz conditions are used as a primary tool, for studying curves in metric spaces in particular.

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1 Basic notions

A *metric space* is a nonempty set M equipped with a distance function $d(x, y)$ defined for $x, y \in M$ such that $d(x, y)$ is a nonnegative real number which is

The present notes are somewhat complementary to [9], and the two can be read in either order.

equal to 0 exactly when $x = y$,

$$(1.1) \quad d(y, x) = d(x, y)$$

for every $x, y \in M$, and

$$(1.2) \quad d(x, z) \leq d(x, y) + d(y, z)$$

for every $x, y, z \in M$. This last condition is known as the *triangle inequality*.

As usual, \mathbf{R} denotes the real line, and the *absolute value* $|r|$ of $r \in \mathbf{R}$ is defined to be r when $r \geq 0$ and $-r$ when $r \leq 0$. It is easy to check that

$$(1.3) \quad |r + t| \leq |r| + |t|$$

for every $r, t \in \mathbf{R}$, which implies that $|r - t|$ is a metric on \mathbf{R} .

Let $(M, d(x, y))$ be a metric space. For each $x \in M$ and $r > 0$, the *open ball* with center x and radius r is defined by

$$(1.4) \quad B(x, r) = \{y \in M : d(x, y) < r\},$$

and the corresponding *closed ball* is defined to be

$$(1.5) \quad \overline{B}(x, r) = \{y \in M : d(x, y) \leq r\}.$$

A set $E \subseteq M$ is said to be *bounded* if E is contained in a ball in M . For any $p, q \in M$ and $r > 0$,

$$(1.6) \quad B(p, r) \subseteq B(q, r + d(p, q))$$

and

$$(1.7) \quad \overline{B}(p, r) \subseteq \overline{B}(q, r + d(p, q)),$$

by the triangle inequality. It follows that a bounded set $E \subseteq M$ is contained in a ball centered at any point in M .

2 Norms on \mathbf{R}^n

For each positive integer n , \mathbf{R}^n is the space of n -tuples $x = (x_1, \dots, x_n)$ of real numbers, i.e., $x_1, \dots, x_n \in \mathbf{R}$. This is a vector space with respect to coordinatewise addition and scalar multiplication by real numbers.

Suppose that $N(x)$ is a function defined on \mathbf{R}^n with values in the nonnegative real numbers. We say that $N(x)$ is a *norm* on \mathbf{R}^n if $N(x) = 0$ exactly when $x = 0$,

$$(2.1) \quad N(rx) = |r|N(x)$$

for every $r \in \mathbf{R}$ and $x \in \mathbf{R}^n$, and

$$(2.2) \quad N(x + y) \leq N(x) + N(y)$$

for every $x, y \in \mathbf{R}^n$. If $N(x)$ is a norm on \mathbf{R}^n , then

$$(2.3) \quad d_N(x, y) = N(x - y)$$

is a metric on \mathbf{R}^n .

The absolute value function is a norm on the real line, and any norm on \mathbf{R} can be expressed as $a|x|$ for some $a > 0$. The standard Euclidean norm on \mathbf{R}^n is defined by

$$(2.4) \quad |x| = \left(\sum_{j=1}^n x_j^2 \right)^{1/2},$$

and determines the standard metric on \mathbf{R}^n . It is not completely obvious that this satisfies the triangle inequality, and one way to show this will be mentioned in the next section. It is much easier to check directly that

$$(2.5) \quad \|x\|_1 = \sum_{j=1}^n |x_j|$$

and

$$(2.6) \quad \|x\|_\infty = \max(|x_1|, \dots, |x_n|)$$

are norms on \mathbf{R}^n . Note that the standard norm may also be denoted $\|x\|_2$.

If $N(x)$ is any norm on \mathbf{R}^n , then

$$(2.7) \quad N(x) \leq \max(N(e_1), \dots, N(e_n)) \|x\|_1,$$

where e_1, \dots, e_n are the standard basis vectors in \mathbf{R}^n , which is to say that the j th coordinate of e_ℓ is equal to 1 when $j = \ell$ and to 0 otherwise. Indeed,

$$(2.8) \quad x = \sum_{j=1}^n x_j e_j,$$

and therefore

$$(2.9) \quad N(x) \leq \sum_{j=1}^n N(e_j) |x_j|.$$

In particular,

$$(2.10) \quad \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$$

for every $x \in \mathbf{R}^n$, since the first inequality holds by inspection. One can also get the second inequality by observing that

$$(2.11) \quad \sum_{j=1}^n x_j^2 \leq \|x\|_1 \|x\|_\infty,$$

since $\|x\|_\infty \leq \|x\|_1$ also holds by inspection. In the other direction, it is easy to see that

$$(2.12) \quad \|x\|_1 \leq n \|x\|_\infty$$

and

$$(2.13) \quad \|x\|_2 \leq \sqrt{n} \|x\|_\infty,$$

and one can use the convexity of $\phi(t) = t^2$ on the real line to show that

$$(2.14) \quad \|x\|_1 \leq \sqrt{n} \|x\|_2.$$

3 Convex sets in \mathbf{R}^n

A set $E \subseteq \mathbf{R}^n$ is said to be *convex* if for every $x, y \in E$ and every real number t , $0 \leq t \leq 1$,

$$(3.1) \quad tx + (1 - t)y \in E.$$

For example, open and closed balls with respect to the metric associated to a norm on \mathbf{R}^n are convex.

Conversely, suppose that $N(x)$ is a nonnegative real-valued function on \mathbf{R}^n that satisfies $N(x) > 0$ when $x \neq 0$ and the homogeneity condition (2.1). If

$$(3.2) \quad B_N = \{x \in \mathbf{R}^n : N(x) \leq 1\}$$

is convex, then N satisfies the triangle inequality (2.2) and hence is a norm. To see this, let $x, y \in \mathbf{R}^n$ be given, and let us check (2.2). The inequality is trivial when $x = 0$ or $y = 0$, and so we may suppose that $x, y \neq 0$. Put

$$(3.3) \quad x' = \frac{x}{N(x)}, \quad y' = \frac{y}{N(y)},$$

which automatically satisfy

$$(3.4) \quad N(x') = N(y') = 1.$$

For $0 \leq t \leq 1$, convexity of B_N implies that

$$(3.5) \quad N(tx' + (1 - t)y') \leq 1.$$

If

$$(3.6) \quad t = \frac{N(x)}{N(x) + N(y)},$$

then

$$(3.7) \quad 1 - t = \frac{N(y)}{N(x) + N(y)}$$

and

$$(3.8) \quad tx' + (1 - t)y' = \frac{x + y}{N(x) + N(y)},$$

which means that (2.2) follows from (3.5).

One can use the convexity of the function $\phi(r) = r^2$ on the real line to show directly that the closed unit ball with respect to the standard Euclidean norm is a convex set in \mathbf{R}^n , and hence that the Euclidean norm satisfies the triangle inequality and is therefore a norm. For each real number p , $1 \leq p < \infty$, put

$$(3.9) \quad \|x\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p}.$$

One can use the convexity of the function $\phi_p(r) = |r|^p$ on the real line to show that the closed unit ball associated to $\|x\|_p$ is a convex set in \mathbf{R}^n , and therefore that $\|x\|_p$ is a norm on \mathbf{R}^n .

By inspection,

$$(3.10) \quad \|x\|_\infty \leq \|x\|_p$$

for every $x \in \mathbf{R}^n$ and $1 \leq p < \infty$. If $1 \leq p < q < \infty$, then

$$(3.11) \quad \sum_{j=1}^n |x_j|^q \leq \left(\sum_{j=1}^n |x_j|^p \right) \|x\|_\infty^{q-p},$$

which implies that

$$(3.12) \quad \|x\|_q \leq \|x\|_p^{p/q} \|x\|_\infty^{1-(p/q)}$$

and thus

$$(3.13) \quad \|x\|_q \leq \|x\|_p$$

for every $x \in \mathbf{R}^n$.

4 Lipschitz conditions, 1

Let $(M_1, d_1(x, y))$, $(M_2, d_2(u, v))$ be metric spaces. A mapping $f : M_1 \rightarrow M_2$ is said to be *Lipschitz* with constant $C \geq 0$ or C -Lipschitz if

$$(4.1) \quad d_2(f(x), f(y)) \leq C d_1(x, y)$$

for every $x, y \in M_1$. More precisely, f is Lipschitz of order 1 if this holds for some $C \geq 0$, and we shall discuss Lipschitz conditions of any order $a > 0$ a bit later. Note that f is Lipschitz with $C = 0$ if and only if f is constant, and that Lipschitz mappings are automatically uniformly continuous.

If M_2 is the real line with the standard metric, then the preceding Lipschitz condition is equivalent to

$$(4.2) \quad f(x) \leq f(y) + C d_1(x, y).$$

This follows by interchanging the order of x and y . In particular,

$$(4.3) \quad f_p(x) = d_1(x, p)$$

is Lipschitz with $C = 1$ for every $p \in M_1$, by the triangle inequality. If f, \tilde{f} are real-valued Lipschitz functions on M_1 with constants C, \tilde{C} , respectively, then $f + \tilde{f}$ is Lipschitz with constant $C + \tilde{C}$. Moreover, $a f$ is Lipschitz with constant $|a| C$ for every $a \in \mathbf{R}$. The product of bounded real-valued Lipschitz functions is also Lipschitz. If $(M_3, d_3(w, z))$ is another metric space, and $f_1 : M_1 \rightarrow M_2$ and $f_2 : M_2 \rightarrow M_3$ are Lipschitz mappings with constants C_1, C_2 , respectively, then the composition $f_2 \circ f_1 : M_1 \rightarrow M_3$ defined by

$$(4.4) \quad (f_2 \circ f_1)(x) = f_2(f_1(x))$$

is Lipschitz with constant $C_1 C_2$.

For any mapping $f : M_1 \rightarrow M_2$ and set $A \subseteq M_1$,

$$(4.5) \quad f(A) = \{f(x) : x \in A\} \subseteq M_2.$$

Let $B_1(x, r)$ and $B_2(p, t)$ be the open balls in M_1, M_2 with centers $x \in M_1, p \in M_2$ and radii $r, t > 0$, respectively. It is easy to see that $f : M_1 \rightarrow M_2$ is Lipschitz with constant $C > 0$ if and only if

$$(4.6) \quad f(B_1(x, r)) \subseteq B_2(f(x), C r)$$

for every $x \in M_1$ and $r > 0$. This is also equivalent to the analogous condition

$$(4.7) \quad f(\overline{B}_1(x, r)) \subseteq \overline{B}_2(f(x), C r)$$

for closed balls. In particular, if A is a bounded set in M_1 , then $f(A)$ is bounded in M_2 .

Suppose that f is a real-valued function on the real line, equipped with the standard metric. If f is differentiable at a point $x \in \mathbf{R}$, and f is C -Lipschitz for some $C \geq 0$, then the derivative $f'(x)$ of f at x satisfies

$$(4.8) \quad |f'(x)| \leq C.$$

This follows from the definition of the derivative. Conversely, if f is differentiable and satisfies (4.8) at every point in \mathbf{R} , then f is C -Lipschitz, by the mean value theorem. Note that $f(x) = |x|$ is 1-Lipschitz on \mathbf{R} and not differentiable at $x = 0$.

5 Lipschitz curves

Let $(M, d(x, y))$ be a metric space, and suppose that a, b are real numbers with $a \leq b$. As usual, the closed interval $[a, b]$ in the real line consists of the $r \in \mathbf{R}$ such that $a \leq r \leq b$. Suppose also that $p : [a, b] \rightarrow M$ is Lipschitz with constant k for some $k \geq 0$. If $\{t_j\}_{j=0}^n$ is a finite sequence of real numbers such that

$$(5.1) \quad a = t_0 < t_1 < \cdots < t_n = b,$$

then

$$(5.2) \quad \sum_{j=1}^n d(p(t_j), p(t_{j-1})) \leq k \sum_{j=1}^n (t_j - t_{j-1}) = k(b - a).$$

This is often described by saying that the length of the curve determined by $p(t)$, $a \leq t \leq b$, has length $\leq k(b - a)$.

Of course, one can use translations on the real line to shift the interval on which a path is defined without changing the Lipschitz constant. One can use affine mappings on \mathbf{R} to change the length of the interval on which a path is defined, with a corresponding change in the Lipschitz constant. The product of the Lipschitz constant and the length of the interval would remain the same.

If $c \in \mathbf{R}$, $c \geq b$, $q : [b, c] \rightarrow M$ is k -Lipschitz, and $p(b) = q(b)$, then the mapping from $[a, c]$ into M defined by combining p and q is k -Lipschitz too. This is easy to verify, directly from the definitions. If the Lipschitz constants for p and q are different, then it may be preferable to rescale the intervals so that the Lipschitz constants are the same. If p is constant on an interval $[a_1, b_1] \subseteq [a, b]$, then one can remove (a_1, b_1) from $[a, b]$ and combine the remaining pieces to get a curve with the same Lipschitz constant on a smaller interval.

6 Minimality

Let $(M, d(x, y))$ be a metric space in which closed and bounded sets are compact. Suppose that $x, y \in M$ can be connected by a Lipschitz curve in M . This means that there is a Lipschitz mapping $p : [0, 1] \rightarrow M$ such that $p(0) = x$ and $p(1) = y$. Using the Arzela-Ascoli theorem, one can show that there is such a path whose Lipschitz constant is as small as possible. For suppose that p_1, p_2, \dots is a sequence of Lipschitz mappings from $[0, 1]$ into M whose Lipschitz constants k_1, k_2, \dots , respectively, converge to the infimum k of the possible Lipschitz constants. By passing to a subsequence, we may suppose that the sequence of mappings converges uniformly on $[0, 1]$. The limiting mapping sends 0 to x and 1 to y , and it is easy to check that it is Lipschitz with constant k . Note that $k \geq d(x, y)$.

7 Affine paths in \mathbf{R}^n

Fix a positive integer n , and consider an affine mapping $p : \mathbf{R} \rightarrow \mathbf{R}^n$, given by $p(r) = u + rv$ for some $u, v \in \mathbf{R}^n$. If N is any norm on \mathbf{R}^n , then p is Lipschitz with constant $N(v)$ with respect to the standard metric on \mathbf{R} and the metric associated to N on \mathbf{R}^n , since

$$(7.1) \quad N(p(r) - p(t)) = |r - t| N(v)$$

for every $r, t \in \mathbf{R}$. For any $a, b \in \mathbf{R}$ with $a \leq b$, the restriction of p to $[a, b]$ is a Lipschitz curve connecting $p(a)$ to $p(b)$ with constant $N(v)$, and $N(v)$ is the smallest possible Lipschitz constant for such a curve on $[a, b]$.

A norm N on \mathbf{R}^n is said to be *strictly convex* if the corresponding closed unit ball B_N as in (3.2) is strictly convex. This means that for every $x, y \in \mathbf{R}^n$ with $N(x) = N(y) = 1$ and $x \neq y$, we have that

$$(7.2) \quad N(tx + (1 - t)y) < 1$$

when $0 < t < 1$. Equivalently, if $w, z \in \mathbf{R}^n$ and

$$(7.3) \quad N(w + z) = N(w) + N(z),$$

then either $w = 0$, $z = 0$, or $z = rw$ for some $r > 0$. This follows from an argument like the one used in Section 3 to show that convexity of B_N implies the triangle inequality for N when N is homogeneous.

One can show that the standard Euclidean norm on \mathbf{R}^n is strictly convex, using strict convexity of the function $\phi(r) = r^2$ on the real line. Similarly, $\|x\|_p$ is strictly convex on \mathbf{R}^n when $1 < p < \infty$, as a consequence of the strict convexity of $\phi_p(r) = |r|^p$. In particular, the absolute value is strictly convex as a norm on \mathbf{R} , if not in the ordinary sense for arbitrary functions, because equality holds in (1.3) only when $r, t \geq 0$ or $r, t \leq 0$. However, $\|x\|_1$ and $\|x\|_\infty$ are not strictly convex norms on \mathbf{R}^n when $n \geq 2$.

Suppose that N is a strictly convex norm on \mathbf{R}^n . If $x, y, z \in \mathbf{R}^n$ satisfy

$$(7.4) \quad d_N(x, z) = d_N(x, y) + d_N(y, z),$$

where d_N is as defined in (2.3), then

$$(7.5) \quad y = r x + (1 - r) z$$

for some $r \in [0, 1]$. If $q : [a, b] \rightarrow \mathbf{R}^n$ is k -Lipschitz and

$$(7.6) \quad N(q(b) - q(a)) = k(b - a),$$

then

$$(7.7) \quad d_N(q(a), q(b)) = d_N(q(a), q(t)) + d_N(q(t), q(b))$$

for every $t \in [a, b]$. One can use this to show that $q(t)$ is affine. This does not work for the norms $\|x\|_1, \|x\|_\infty$ on \mathbf{R}^n when $n \geq 2$. For example, there is a 1-Lipschitz path from $[0, 2]$ into \mathbf{R}^2 equipped with the norm $\|x\|_1$ that connects $(0, 0)$ to $(1, 1)$ by following the horizontal segment to $(1, 0)$ and then the vertical segment to $(1, 1)$. If $\phi : [0, 1] \rightarrow \mathbf{R}$ is any 1-Lipschitz function with respect to the standard metric on the real line which satisfies $\phi(0) = \phi(1) = 0$, then $\Phi(t) = (t, \phi(t))$ is a 1-Lipschitz mapping from $[0, 1]$ into \mathbf{R}^2 equipped with the norm $\|x\|_\infty$ that connects $(0, 0)$ to $(1, 0)$.

8 C^1 paths in \mathbf{R}^n

Let N be a norm on \mathbf{R}^n . As in Section 4, the triangle inequality implies that N is 1-Lipschitz with respect to the associated metric d_N . As in Section 2, one can show that N is less than or equal to a constant multiple of the standard Euclidean norm on \mathbf{R}^n . It follows that N is also a Lipschitz function with respect to the standard metric on \mathbf{R}^n .

Let $p : [a, b] \rightarrow \mathbf{R}^n$ be a continuously-differentiable curve with derivative $p'(t)$. This implies that $N(p'(t))$ is a continuous function on $[a, b]$. By the fundamental theorem of calculus,

$$(8.1) \quad p(t) - p(r) = \int_r^t p'(u) du$$

when $a \leq r \leq t \leq b$. Hence

$$(8.2) \quad N(p(t) - p(r)) \leq \int_r^t N(p'(u)) du,$$

using an extension of the triangle inequality from sums to integrals. If

$$(8.3) \quad N(p'(u)) \leq k$$

for every $u \in [a, b]$, then it follows that p is k -Lipschitz with respect to the metric associated to N on \mathbf{R}^n .

Alternatively, let $\epsilon > 0$ be given. For each $r \in [a, b]$,

$$(8.4) \quad p(t) - p(r) - p'(r)(t - r)$$

is ϵ -Lipschitz as a function of t on sufficiently small neighborhoods of r in $[a, b]$, since p is continuously-differentiable. Under the hypothesis (8.3), we get that p is $(k + \epsilon)$ -Lipschitz with respect to N on sufficiently small neighborhoods of every point in $[a, b]$. One can use this to show that p is $(k + \epsilon)$ -Lipschitz on $[a, b]$, and therefore k -Lipschitz because $\epsilon > 0$ is arbitrary. Note that (8.3) holds when p is k -Lipschitz with respect to N on $[a, b]$.

In order for the product of the Lipschitz constant and the length of the parameter interval to be as small as possible, it would be nice to have $N(p')$ constant on $[a, b]$. As in the classical situation, one can try to get this by reparameterizing p . This is easy to do when $p'(t) \neq 0$ for every $t \in [a, b]$. Specifically,

$$(8.5) \quad \phi(t) = \int_a^t N(p'(u)) du$$

is a continuously-differentiable function on $[a, b]$ with

$$(8.6) \quad \phi'(t) = N(p'(t)) > 0$$

for each $t \in [a, b]$. If $q = p \circ \phi^{-1}$, then

$$(8.7) \quad N(q'(r)) = 1$$

when $\phi(a) \leq r \leq \phi(b)$.

9 Lipschitz conditions, 2

Let $(M_1, d_1(x, y))$ and $(M_2, d_2(u, v))$ be metric spaces. A mapping $f : M_1 \rightarrow M_2$ is said to be *Lipschitz of order $\alpha > 0$* with constant $C \geq 0$ if

$$(9.1) \quad d_2(f(x), f(y)) \leq C d_1(x, y)^\alpha$$

for every $x, y \in M_1$. As before, this holds with $C = 0$ if and only if f is constant, and Lipschitz mappings of any order are uniformly continuous. If a real-valued function on the real line is Lipschitz of order $\alpha > 1$, then it is constant, because it has derivative 0, although one could also show this more directly. It follows that a Lipschitz mapping of order $\alpha > 1$ from an interval in the real line into any metric space is constant as well, by composing with real-valued Lipschitz functions of order 1 on the range, such as the distance to a fixed point.

Suppose that $0 < \beta < 1$. If $r, t \geq 0$, then

$$(9.2) \quad \max(r, t) \leq (r^\beta + t^\beta)^{1/\beta}.$$

Therefore

$$(9.3) \quad r + t \leq \max(r, t)^{1-\beta} (r^\beta + t^\beta) \leq (r^\beta + t^\beta)^{1/\beta},$$

or equivalently

$$(9.4) \quad (r + t)^\beta \leq r^\beta + t^\beta.$$

This is also very easy to check algebraically when $\beta = 1/2$, for instance.

If $(M, d(w, z))$ is a metric space, then it follows that $d(w, z)^\beta$ is a metric on M too when $0 < \beta < 1$. This does not work when $\beta > 1$, even for the real line. Observe that $f : M_1 \rightarrow M_2$ is Lipschitz of order α with respect to $d_1(x, y)$ on M_1 if and only if f is Lipschitz of order α/β with respect to $d_1(x, y)^\beta$, keeping $d_2(u, v)$ fixed on M_2 . Similarly, f is Lipschitz of order α with respect to $d_2(u, v)$ on M_2 if and only if f is Lipschitz of order $\alpha\beta$ with respect to $d_2(u, v)^\beta$ on M_2 , keeping $d_1(x, y)$ fixed on M_1 .

A curve $p : [a, b] \rightarrow M$ in a metric space $(M, d(w, z))$ parameterized by a Lipschitz mapping of order $\alpha < 1$ can be quite different from the case where $\alpha = 1$. The length of p can be infinite, and moreover $p([a, b])$ can be fractal. This includes common examples of snowflake curves in the plane. Instead one can show that the α -dimensional Hausdorff measure of $p([a, b])$ is finite.

References

- [1] R. Beals, *Analysis: An Introduction*, Cambridge University Press, 2004.
- [2] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, 1986.
- [3] R. Goldberg, *Methods of Real Analysis*, 2nd edition, Wiley, 1976.
- [4] S. Krantz, *Real Analysis and Foundations*, 2nd edition, Chapman & Hall / CRC, 2005.
- [5] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature*, European Mathematical Society, 2005.
- [6] M. Rosenlicht, *Introduction to Analysis*, Dover, 1986.
- [7] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.
- [8] S. Semmes, *What is a metric space?*, arXiv:0709.1676 [math.MG].
- [9] S. Semmes, *An introduction to the geometry of metric spaces*, arXiv:0709.4239 [math.MG].